

Transport coefficients of non-Newtonian fluid and causal dissipative hydrodynamics

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A formula to calculate the transport coefficients of the causal dissipative hydrodynamics is derived by using the projection operator method (Mori-Zwanzig formalism) [T. Koide, Phys. Rev. E **75**, 060103(R) (2007)]. This is an extension of the Green-Kubo-Nakano (GKN) formula to the case of non-Newtonian fluids, which is the essential factor to preserve the relativistic causality in relativistic dissipative hydrodynamics. This formula is the generalization of the GKN formula in the sense that it can reproduce the GKN formula in a certain limit. In this work, we extend the previous work so as to apply to more general situations.

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I. INTRODUCTION

Hydrodynamic models have been extensively applied to analyze the collective aspects of relativistic heavy-ion collisions. These analyses have mainly been done so far for ideal fluids [1]. The effect of dissipation (viscosity and heat conduction) to this problem has started only recently and it is less well understood yet. One of the reasons for this is that, in addition to technical difficulties in numerical implementations, a relativistic extension of the dissipative hydrodynamics is not trivial at all conceptually [2–10]. A naive covariant extension of the Navier-Stokes equation leads to the problem of relativistic acausality and instabilities of the theory [9,11–13].

An essential factor to solve this problem is to introduce a memory effect with a finite relaxation time in the definition of irreversible currents [7,10,14,15]. An important point here is that, with the presence of memory effects, the fluid becomes non-Newtonian, that is, the irreversible current is not simply proportional to the thermodynamical forces.

This raises several serious questions in applying the causal dissipative hydrodynamics to various phenomena at relativistic energies. The crucial one, we will focus in this paper, is that we cannot use the Green-Kubo-Nakano (GKN) formula to calculate the transport coefficients, because the derivation depends on the Newtonian property of the fluid. See the discussion in Appendix A. To obtain the transport coefficients of the causal dissipative hydrodynamics, a new formulation should be developed.

One possible approach to obtain the transport coefficients in the presence of memory effects is the so-called projection operator method (POM). The POM was originally proposed to obtain master equations and generalized Langevin equations from microscopic dynamics by implementing systematic coarse-grainings in terms of projection operators for macroscopic variables [16–23]. It is also known that the POM is useful to obtain the microscopic expressions of various transport coefficients [20–22,24]. In the POM, the transport coefficients are related to the memory function of the generalized Langevin equation. Except for trivial cases, it is very difficult to evaluate the memory function exactly and some appropriate approximations are needed. A common method is to neglect a part of the projection operator in the memory function (as explained later). In the following, we

call such an approximation as the Q approximation. It is known that the formula for the transport coefficients obtained with the Q approximation in the POM are equivalent to those of the GKN formula [20–22,24]. It is further known that the coarse-grained equation of a conserved quantity obtained by the POM with the Q approximation becomes a usual diffusion equation [20–22,25]. That is, the use of the Q approximation in the POM leads to the behaviors of Newtonian fluids.

Recently, one of the present authors discussed the coarse-graining procedure in the POM without using the Q approximation [24–26]. There, it was shown that the equation for a conserved number becomes a telegraph-type equation when the Q approximation is not introduced [25]. Note that the telegraphic equation is derived when a memory effect is introduced in a diffusion equation [25]. This indicates that we can apply this method to construct the causal dissipative hydrodynamics in the POM, defining the microscopic expressions of the transport coefficients for non-Newtonian fluids in a consistent manner. Following this idea, new formulas of the transport coefficients for the causal dissipative hydrodynamics have been derived [24]. This new formula differs in an essential way from those obtained using the GKN formula with the Newtonian case, although it can reproduce the GKN formula under a limit where the Q approximation is valid.

In this paper, we present a more detailed version of the work of Ref. [24] and derive more general expressions of transport coefficients, in particular, the shear viscosity of the causal dissipative hydrodynamics. This paper is organized as follows. In Sec. II, for the sake of later convenience, we review briefly the projection operator method to derive the generalized Langevin equation. In Sec. III, the so-called Mori projection operator is introduced. We calculate explicitly the memory function in Sec. IV. This result is the generalization of the formula obtained in Ref. [24] and one of the main results of this paper. By using this general expression, we define the causal shear viscosity coefficient and the relaxation time in Sec. V. The result of this section is completely same as that of Ref. [24]. The relation between our formula and the GKN formula is discussed in Sec. VI. In Sec. VII, we apply the result to an exactly solvable model to confirm the validity of our exact expression of the memory function. In Sec. VIII, we reinvestigate the result of the Sec. V and propose another possible definition of the causal shear viscosity coefficient. The Sec. IX is devoted to concluding remarks.

II. PROJECTION OPERATOR METHOD

It should be emphasized that the projection operator method (POM) was firstly proposed by Nakajima [16], although it is often referred as the Mori-Zwanzig formalism due to the extensive use and developments done by these authors. This approach has been studied so far in various contexts of physics and chemistry [20–22]. In particular, Mori introduced the so-called Mori projection operator to describe the dynamics near thermal equilibrium and derived a generalized Langevin equation from microscopic models [18]. The generalized Langevin equation (the Mori equation) gives the basis of the various development of statistical physics. Kawasaki, for example, developed the mode coupling theory which describes the dynamical critical phenomena by using the technique of the generalized Langevin equation [27]. The mode coupling theory is recently used to discuss the glass dynamics [28]. It is considered that the POM is a promising method to establish a new coarse-grained dynamics such as the dynamical density functional theory [29,30]. The formulation of the projection operator method has been polished up by several authors [31–38]. The derivation discussed here follows [36].

In a quantum mechanical system, the time evolution of an operator is governed by the Heisenberg equation of motion

$$\frac{d}{dt}O(t) = i[H, O(t)] = iLO(t) \quad (1)$$

$$\rightarrow O(t) = e^{iL(t-t_0)}O(t_0), \quad (2)$$

where L is the Liouville operator and t_0 is an initial time at which we prepare an initial state. In the following, we set $t_0=0$. We consider here an isolated system so that the Hamiltonian is independent of time. Note that Eq. (2) is also valid for classical cases provided that the commutator of the Liouville operator is interpreted as the Poisson bracket.

In order to derive coarse-grained equations such as hydrodynamical equation of motion from a microscopic theory, we should construct a closed system of equations expressed only by those variables with macroscopic properties of the system. However, the Heisenberg equation of motion contains the information not only of gross variables associated with macroscopic (hydrodynamic) time scales, but also of microscopic variables. In the POM, the latter variables are projected out by introducing an appropriate projection operator P (to be specified later). We denote its complementary operator by $Q(=1-P)$. They should satisfy,

$$P^2 = P, \quad (3)$$

$$PQ = QP = 0. \quad (4)$$

Here, the projection operators are time-independent. To describe real nonequilibrium processes, in general, the projection operator should be time dependent. However, for the purpose of the present paper, simple time-independent projection operators are suffice as the definition of the transport coefficients of the relativistic dissipative fluid.

From Eq. (2), one can see that the time dependence of operators is determined by e^{iLt} . This operator obeys the following differential equations:

$$\frac{d}{dt}e^{iLt} = e^{iLt}iL = e^{iLt}(P+Q)iL. \quad (5)$$

Multiplying the operator Q from the right, we have

$$\frac{d}{dt}e^{iLt}Q = e^{iLt}PiLQ + e^{iLt}QiLQ. \quad (6)$$

Equation (6) can be solved for $e^{iLt}Q$,

$$e^{iLt}Q = Qe^{iLQt} + \int_0^t d\tau e^{iL\tau}PiLQe^{iLQ(t-\tau)}. \quad (7)$$

Substituting Eq. (7) into the last term in Eq. (5) and operating $O(0)$ from the right, we obtain the so-called time-convolution (TC) equation

$$\begin{aligned} \frac{d}{dt}O(t) = & e^{iLt}PiLO(0) + \int_0^t d\tau e^{iL(t-\tau)}PiLQe^{iLQ\tau}iLO(0) \\ & + Qe^{iLQt}iLO(0). \end{aligned} \quad (8)$$

The first term on the right-hand side of the equation is called the streaming term and usually corresponds to collective modes such as plasma wave, spin wave and so on. The second term is the memory term that causes dissipation. The third term represents the noise term. The second term and third terms are related through the fluctuation-dissipation theorem of second kind, which will be discussed later.

Discussion of this section has been done in the Heisenberg picture and the generalized Langevin equation (8) is derived. We can develop the similar discussion in the Schrödinger picture and obtain master equations. As we pointed out, to discuss more complex nonequilibrium processes, we have to change the basis of the projection with time. Then, the projection operator is explicitly time-dependent. As for the operation of the time-dependent projection operator, see Ref. [37], and references therein.

It is also possible to derive another form of the generalized Langevin equation, which is called the time-convolutionless (TCL) equation. There are some cases where we cannot implement the Markov approximation in the TC equation. The ϕ^4 theory is one of the examples, and the Markov equation is derived from the TCL equation. See Ref. [38] for details.

III. MORI PROJECTION OPERATOR

In the above derivation of the TC equation, we have not specified the projection operator P . As a matter of fact, there are many possible projection operators that extract the slowly varying components from dynamics. Suppose that the macroscopic dynamics can be described by the time evolutions of n -gross variables. Then we have to define the projection operator to project any time evolution onto the space spanned by these n -gross variables. For example, in the case of usual hydrodynamics, the time evolutions are described by

the dynamics of the energy density, velocity field and number density. Then we have five variables which form a complete set for hydrodynamics (two scalar fields and one vector field).

Strictly speaking, there is no general criterion to prepare a complete set of gross variables. It is, however, suggested that there are three candidates for gross variables [18,27]: (i) Order parameters (if there exists phase transitions), (ii) density variables of conserved quantities, and (iii) their products. Once we find out a complete set of the gross variables, any macroscopic variables should be approximately given by a linear combination of these gross variables (see below). We can use the Mori projection operator to implement this coarse graining.

Let us represents a set of gross variables by a n -dimensional vector

$$\mathbf{A}^T = (A_1, A_2, \dots, A_n). \quad (9)$$

Then, the Mori projection operator P is defined as

$$PO = \sum_{i=1}^n c_i A_i, \quad (10)$$

for an arbitrary operator O , where the coefficient c_i is given by

$$c_i = \sum_{j=1}^n (O, A_j^\dagger)(A, A^\dagger)_{ji}^{-1}. \quad (11)$$

The inner product is Kubo's canonical correlation

$$(X, Y) = \int_0^\beta \frac{d\lambda}{\beta} \text{Tr}[\rho e^{\lambda H} X e^{-\lambda H} Y], \quad (12)$$

where $\rho = e^{-\beta H} / \text{Tr}[e^{-\beta H}]$ with the temperature β^{-1} . The inverse of the canonical correlation is defined by

$$\sum_j (A, A)_{ij}^{-1} (A_j, A_k) = \delta_{i,k}. \quad (13)$$

As for the physical meaning of the Mori projection operator, see, for example, Refs. [18,26].

IV. THE EXACT EXPRESSION OF THE MEMORY FUNCTION

Substituting the Mori projection operator into Eq. (8), the TC equation is reexpressed as follows:

$$\frac{\partial}{\partial t} \mathbf{A}(t) = i\Delta \mathbf{A}(t) - \int_0^t d\tau \Xi(\tau) \mathbf{A}(t - \tau) + \xi(t), \quad (14)$$

where

$$i\Delta = \sum_k (iLA_{i,A_k^\dagger})(A, A^\dagger)_{kj}^{-1}, \quad (15)$$

$$\Xi_{ij}(t) = -\theta(t) \sum_k (iLQe^{iLQt}iLA_{i,A_k^\dagger})(A, A^\dagger)_{kj}^{-1}, \quad (16)$$

$$\xi_i(t) = Qe^{iLQt}iLA_i. \quad (17)$$

The TC equation using the Mori projection operator is called the Mori equation.

The important information of dissipation is given by the memory function $\Xi(t)$. As a matter of fact, transport coefficients are defined by it. However, the calculation of the memory function is not simple because the expression of the memory function has the projection operator Q [see Eq. (16)]. As a matter of fact, the projection operator Q is approximately replaced with 1 to estimate the memory function in many textbooks [20–22],

$$\Xi(t) \approx -\theta(t) \sum_k (iLe^{iL\tau}iLA_{i,A_k^\dagger})(A, A^\dagger)_{kj}^{-1}. \quad (18)$$

As mentioned in the Introduction, we call this procedure the Q approximation.

Recently, Okada *et al.* calculated the memory function of the Ising model and found that the memory function can be expressed in terms of the combination of the usual time correlation functions [39]. Afterwards, Koide applied the same procedure to microscopic models and discussed the effect of the Q approximation [25,26]. The coarse-grained dynamics of conserved quantities of the model are, usually, considered to be given by the diffusion equation. As a matter of fact, when we apply the Q approximation, we can derive the diffusion equation in the model. When we do not apply the Q approximation, however, the coarse-grained dynamics is given by the telegraph-type equation instead of the diffusion equation [25]. More interestingly, if the model has a conserved quantity, we can derive the sum rule associated with the conserved law. It was shown that the telegraph-type equation derived with the memory effect is consistent with the sum rule, while the diffusion equation (Q approximation) breaks it [25,40].

The same idea is used to define the transport coefficients of the causal dissipative hydrodynamics by using the simplest Mori projection operator that is defined with only one gross variable [24]. In this section, we extend the discussion of Ref. [24] to more complex cases where the Mori projection operator is defined with n -gross variables.

To calculate the coarse-grained time-evolution operator, we introduce the following operator:

$$\mathcal{B}(t) = 1 + \sum_{n=1}^{\infty} (-i)^n \int_0^t dt_1 \cdots \int_0^{t_{n-1}} dt_n \check{L}^P(t_1) \check{L}^P(t_2) \cdots \check{L}^P(t_n), \quad (19)$$

$$(20)$$

with

$$\check{L}^P(t) \equiv e^{-iLt} P L e^{iLt}. \quad (21)$$

Then, the matrix $\Xi(t)$ is rewritten as

$$\begin{aligned}
\Xi_{ij}(t) &= -\theta(t)[iL e^{iLt} \mathcal{B}(t) Q iL A_i A_j^\dagger](A_j, A_i^\dagger)^{-1} \\
&= -\theta(t) \left\{ [\ddot{\mathbf{X}}(t)]_{ij} - [\dot{\mathbf{X}}(0) \dot{\mathbf{X}}(t)]_{ij} \right. \\
&\quad + \sum_{n=1}^{\infty} (-1)^n \int_0^t dt_1 \cdots \int_0^{t_{n-1}} dt_n [\ddot{\mathbf{X}}(t_n) \dot{\mathbf{X}}(t_{n-1} - t_n) \cdots \\
&\quad \times \dot{\mathbf{X}}(t_1 - t_2) \dot{\mathbf{X}}(t - t_1)]_{ij} \\
&\quad \left. - \sum_{n=1}^{\infty} (-1)^n \int_0^t dt_1 \cdots \int_0^{t_{n-1}} dt_n [\dot{\mathbf{X}}(0) \dot{\mathbf{X}}(t_n) \right. \\
&\quad \left. \times \dot{\mathbf{X}}(t_{n-1} - t_n) \cdots \dot{\mathbf{X}}(t_1 - t_2) \dot{\mathbf{X}}(t - t_1)]_{ij} \right\}. \quad (22)
\end{aligned}$$

See Appendix B for detailed discussion. By using the Laplace transform, the above expression can be rewritten as the following simple form:

$$\Xi^L(s) = -\ddot{\mathbf{X}}^L(s) \frac{1}{1 + \dot{\mathbf{X}}^L(s)} + \dot{\mathbf{X}}(0) \dot{\mathbf{X}}^L(s) \frac{1}{1 + \dot{\mathbf{X}}^L(s)}, \quad (23)$$

where the functions $\dot{\mathbf{X}}^L(s)$ and $\ddot{\mathbf{X}}^L(s)$ are given by the Laplace transform of the following correlation functions:

$$\begin{aligned}
\dot{\mathbf{X}}_{ij}(t) &= \sum_k [iL A_i(t), A_k^\dagger](A, A^\dagger)_{kj}^{-1}, \\
\ddot{\mathbf{X}}_{ij}(t) &= \sum_k [(iL)^2 A_i(t), A_k^\dagger](A, A^\dagger)_{kj}^{-1}. \quad (24)
\end{aligned}$$

This is the exact expression of the memory function without using the Q approximation. One can see that if we set $n=1$, the expression (23) reproduces the result of the previous result of Ref. [24]. The expression of the transport coefficients are derived by employing approximations to this memory function, as will see in the next section.

V. SHEAR VISCOSITY OF CAUSAL DISSIPATIVE HYDRODYNAMICS IN $N=1$ FORM

We apply the formula to define the shear viscosity coefficient of causal dissipative hydrodynamics. For simplicity, we consider a particular case of shear flow, where the fluid velocity points in the x direction and varies spatially in the y direction [22,24]. Then, the energy-momentum tensor obeys the following equation of motion:

$$\frac{\partial}{\partial t} T^{0x}(y, t) = -\frac{\partial}{\partial y} T^{yx}(y, t) = -\frac{\partial}{\partial y} \pi^{yx}(y, t), \quad (25)$$

where $\pi^{\alpha\beta}$ is the traceless part of the energy-momentum tensor

$$\pi^{kl} = \left(\delta_i^k \delta_j^l - \frac{1}{3} \delta^{kl} \delta_{ij} \right) T^{ij} \quad (i, j, k, l = 1, 2, 3). \quad (26)$$

To define the shear viscosity coefficient, we introduce the Fourier transform of $T^{0x}(y, t)$, and set $O(0) = T^{0x}(k_y, 0)$. And

following the usual derivation of the shear viscosity coefficient of the GKN formula, we choose $T^{0x}(k_y, 0)$ as a unique gross variable [22,24]. In this case, the Mori projection operator is defined by

$$PO = (O, T^{0x}(-k_y, 0))(T^{0x}(k_y, 0), T^{0x}(-k_y, 0))^{-1} T^{0x}(k_y, 0), \quad (27)$$

where O is an arbitrary operator. We will discuss later a more involved case where we need two gross variables to define the Mori projection operator.

Then, the TC equation (8) is given by

$$\begin{aligned}
\frac{\partial}{\partial t} T^{0x}(k_y, t) &= i\Delta(k_y) T^{0x}(k_y, t) - \int_0^t d\tau \Xi(k_y, t - \tau) T^{0x}(k_y, \tau) \\
&\quad + \xi(k_y, t), \quad (28)
\end{aligned}$$

where

$$i\Delta(k_y) = (iL T^{0x}(k_y, 0), T^{0x}(-k_y, 0))(T^{0x}(k_y, 0), T^{0x}(-k_y, 0))^{-1}, \quad (29)$$

$$\Xi(k_y, t) = \frac{1}{2\pi i} \int_{\text{Br}} \Xi^L(k_y, s) e^{st} ds, \quad (30)$$

$$\xi(k_y, t) = Q e^{iLQt} iL T^{0x}(k_y, t). \quad (31)$$

The Laplace transform of the memory function $\Xi^L(k_y, s)$ is given by

$$\Xi^L(k_y, s) = -\dot{\mathbf{X}}^L(k_y, s) \frac{1}{1 + \dot{\mathbf{X}}^L(k_y, s)}, \quad (32)$$

where

$$\begin{aligned}
\dot{\mathbf{X}}(k_y, t) &= (iL T^{0x}(k_y, t), T^{0x}(-k_y, 0))(T^{0x}(k_y, 0), T^{0x}(-k_y, 0))^{-1}, \\
\ddot{\mathbf{X}}(k_y, t) &= ((iL)^2 T^{0x}(k_y, t), T^{0x}(-k_y, 0)) \\
&\quad \times (T^{0x}(k_y, 0), T^{0x}(-k_y, 0))^{-1}. \quad (33)
\end{aligned}$$

In this derivation, we used

$$\dot{\mathbf{X}}(0) = (iL T^{0x}(k_y, 0), T^{0x}(-k_y, 0)) = 0. \quad (34)$$

So far, everything is exact formally. Now we carry out the coarse grainings of the time scale to break the time-reversal symmetry. For this purpose, first of all, we separate the memory function into the two terms as follows [24–26]:

$$\begin{aligned}
\frac{\partial}{\partial t} T^{0x}(k_y, t) &= -\int_0^t d\tau \Omega^2(k_y, t - \tau) T^{0x}(k_y, \tau) \\
&\quad - \int_0^t d\tau \Phi(k_y, t - \tau) T^{0x}(k_y, \tau). \quad (35)
\end{aligned}$$

Here, we dropped the noise term. The frequency function and the renormalized memory function are defined by

$$\Omega^2(k_y, t) = i \int \frac{d\omega}{2\pi} \text{Im}[\Xi^L(k_y, -i\omega + \epsilon)] e^{-i\omega t}, \quad (36)$$

$$\Phi(k_y, t) = \int \frac{d\omega}{2\pi} \operatorname{Re}[\Xi^L(k_y, -i\omega + \epsilon)]e^{-i\omega t}, \quad (37)$$

respectively.

To introduce the coarse graining in time, we have to know the temporal behavior of the two functions. The behavior of the two functions have been investigated for some special cases, such as the chiral order parameter in the Nambu-Jona-Lasinio model [26], exactly solvable model of many harmonic oscillators [26] and the nonrelativistic model with a conserved density [25]. For all these cases, the two functions exhibit common properties; the frequency function converges to a finite value and the renormalized memory function vanishes at late time. Inspired by these examples, we introduce an important assumption that these features for the temporal behavior of the two functions are valid in general. That is, the renormalized memory function relaxes rapidly and vanishes at large t , while the frequency function converges to a finite value after short time evolution. In principle, the validity of the assumption should be checked for more general examples by implementing numerical calculations. Once we accept the above assumption, we may introduce the following *Ansatzse* for the memory functions incorporating these basic features essentially [26]:

$$\Omega^2(k_y, t) \rightarrow D_{k_y} k_y^2, \quad \Phi(k_y, t) \rightarrow \frac{2}{\tau_{k_y}} \delta(t), \quad (38)$$

where

$$D_{k_y} = \frac{1}{k_y^2} \lim_{t \rightarrow \infty} \Omega^2(k_y, t), \quad (39)$$

$$\frac{1}{\tau_{k_y}} = \int_0^\infty dt \Phi(k_y, t). \quad (40)$$

The factor k_y^2 is introduced for the later convenience [see Eq. (43)]. The above ansatz are shown to be consistent with the final value theorem of the Laplace transformation [24–26]. That is, when the renormalized memory function converges to zero at late time, its Laplace transform $\Phi^L(k_y, s)$ should satisfy [41]

$$\lim_{t \rightarrow \infty} \Phi(k_y, t) = \lim_{s \rightarrow 0} s \Phi^L(k_y, s) = 0. \quad (41)$$

Similarly, for the frequency function

$$D_{k_y} k_y^2 = \lim_{t \rightarrow \infty} \Omega^2(k_y, t) = \lim_{s \rightarrow 0} s (\Omega^2)^L(k_y, s). \quad (42)$$

Using these expressions, we have the equation for the energy momentum tensor component

$$\frac{\partial}{\partial t} T^{0x}(k_y, t) = -D_{k_y} k_y^2 \int_0^t d\tau w u^x(k_y) - \frac{1}{\tau_{k_y}} T^{0x}(k_y, t). \quad (43)$$

Here, we expressed the x component of the fluid velocity as

$$u^x(k_y) = T^{0x}(k_y)/w, \quad (44)$$

where w is an enthalpy density [24].

On the other hand, as pointed out before, the time evolution of the energy-momentum tensor in causal dissipative hydrodynamics is given by a kind of the telegraph equation. In particular, in the special case discussed here (the fluid velocity points in the x direction and varies spatially in the y direction), the linearized equation of the causal dissipative hydrodynamics is given by the following telegraph equation [7,10]:

$$\frac{\partial^2}{\partial t^2} T^{0x} + \frac{1}{\tau} \frac{\partial}{\partial t} T^{0x} + \frac{\eta^{NN}}{2\tau} \frac{\partial^2}{\partial y^2} u^x = 0. \quad (45)$$

This equation defines the causal shear viscosity coefficient η^{NN} and corresponding relaxation time τ . By comparing Eq. (43) with Eq. (45), we obtain the expression for the causal shear viscosity coefficient and the respective relaxation time as

$$\eta^{NN} = \lim_{k_y \rightarrow 0} 2w D_{k_y} \tau, \quad (46)$$

$$\tau = \lim_{k_y \rightarrow 0} \tau_{k_y}, \quad (47)$$

which are the results obtained in Ref. [24]. In this derivation, it is assumed that the projection operator is defined by only one gross variable. We will reconsider this derivation in Sec. VIII.

VI. GREEN-KUBO-NAKANO FORMULA

We obtained an expression of the causal shear viscosity coefficient for non-Newtonian fluids. On the other hand, it is well known that the shear viscosity coefficient of Newtonian fluids is given by the GKN formula. In this section, we discuss the relation between our formula and the GKN formula [24].

As was mentioned before, it is known that, when we apply the Q approximation, the GKN formula is reproduced in the projection operator method [42]. When we apply the Q approximation, the memory function is given by [20,22],

$$\Xi^L(k_y, s) \approx -\ddot{X}^L(k_y, s). \quad (48)$$

On the other hand, when the correlation function $\dot{X}^L(k_y, s)$ is very small, the memory function (32) is then expanded as follows:

$$\Xi^L(k_y, s) = -\ddot{X}^L(k_y, s) + \dot{X}^L(k_y, s)\dot{X}^L(k_y, s) - \dots. \quad (49)$$

That is, the correlation function $\dot{X}^L(k_y, s)$ represents the correction to the Q approximation.

As a matter of fact, in the Q approximation, we can derive the relativistic Navier-Stokes equation and then our formula reproduces the GKN formula of the shear viscosity coefficient. First of all, the correlation function $\ddot{X}(k_y, t)$ is rewritten as

$$\begin{aligned}\ddot{X}^L(\mathbf{k},s) &= \int_0^\infty dt e^{-st} \theta(t) \frac{1}{10} \int d^3\mathbf{x} d^3\mathbf{x}_1 e^{-i\mathbf{k}\cdot\mathbf{x}} \mathbf{k}^2 (\pi^{\alpha\beta}(\mathbf{x},t), \pi_{\alpha\beta}(\mathbf{x}_1,0)) (T^{0x}(\mathbf{x}_1,0), T^{0x}(\mathbf{0},0))^{-1} \\ &= -\frac{1}{10\beta} \int_0^\infty dt \int d^3\mathbf{x} d^3\mathbf{x}_1 e^{-st-i\mathbf{k}\cdot\mathbf{x}} \mathbf{k}^2 \int_0^\infty d\tau \langle \pi^{\alpha\beta}(\mathbf{x},\tau) \pi_{\alpha\beta}(\mathbf{x}_1,0) \rangle_{\text{ret}} (T^{0x}(\mathbf{x}_1,0), T^{0x}(\mathbf{0},0))^{-1},\end{aligned}\quad (50)$$

where

$$\langle \pi^{\alpha\beta}(\mathbf{x},t) \pi_{\alpha\beta}(\mathbf{x}_1,s) \rangle_{\text{ret}} = -i\theta(t-s) \langle [\pi^{\alpha\beta}(\mathbf{x},t), \pi_{\alpha\beta}(\mathbf{x}_1,s)] \rangle_{\text{eq}}. \quad (51)$$

In this derivation, we used the relation [43]

$$(\pi_{\mu\nu}, \pi_{\rho\sigma}) = \frac{L_\pi}{2} \left(\Delta_{\mu\rho} \Delta_{\nu\sigma} - \Delta_{\mu\sigma} \Delta_{\nu\rho} - \frac{2}{3} \Delta_{\mu\nu} \Delta_{\rho\sigma} \right), \quad (52)$$

where L_π is a scalar function and

$$\Delta_{\mu\nu} = g_{\mu\nu} - u_\mu u_\nu, \quad (53)$$

with u_μ the four-velocity of the fluid in the Landau frame. The correlation function $\ddot{X}^L(\mathbf{k}, -i\omega + \epsilon)$ is real in the low momentum limit. Then, the frequency function vanishes and the equation of the energy-momentum tensor (43) is given by the linearized relativistic Navier-Stokes equation

$$\frac{\partial}{\partial t} T^{0x} - \eta^{\text{NS}} k_y^2 T^{0x} = 0, \quad (54)$$

where the Navier-Stokes shear viscosity coefficient is

$$\lim_{\mathbf{k} \rightarrow 0} \frac{1}{\tau_{\mathbf{k}}} = -\eta^{\text{NS}} \mathbf{k}^2. \quad (55)$$

By using the expression of $\tau_{\mathbf{k}}$, the Navier-Stokes shear viscosity coefficient is expressed by using the time correlation function as follows:

$$\begin{aligned}\eta^{\text{NS}} &= \int_{-\infty}^\infty dt \theta(t) \frac{1}{10} \int d^3\mathbf{x}_1 (\pi^{\alpha\beta}(\mathbf{x},t), \pi_{\alpha\beta}(\mathbf{x}_1,0)) \\ &\quad \times (T^{0x}(\mathbf{x}_1,0), T^{0x}(\mathbf{0},0))^{-1}.\end{aligned}\quad (56)$$

Except for the normalization factor $(T^{0x}(\mathbf{x}_1,0), T^{0x}(\mathbf{0},0))^{-1}$, this expression is nothing but the GKN formula of the shear viscosity coefficient [42,43]. That is, our new formula can reproduce the result of the GKN formula when the correlation function $\dot{X}^L(k_y, s)$ disappears in the low momentum limit. In this sense, our formula is the generalization of the GKN formula.

As was pointed out, the vanishing $\dot{X}^L(k_y, s)$ corresponds to the Q approximation. So far, because the exact expression of the memory function (23) was not known, we could not discuss whether the Q approximation is applicable in the low momentum limit or not. Now the validity of the Q approximation can be quantitatively estimated by calculating the correlation function $\dot{X}^L(k_y, s)$. In fact, it is already known that there are examples where the Q approximation cannot be applicable [25,26].

VII. MODE-COUPLING THEORY OF DENSITY FLUCTUATIONS

So far, we discussed the simplest case where the system has only one gross variable. In this section, we will consider a more complex case where we need two gross variables to define the Mori projection operator. Such a situation will occur, for example, in a glass dynamics [28]. Glass is a high density system and a particle is thickly surrounded by other particles. The energy and momentum of the particles are continuously exchanged by collisions. However, it is difficult for particles to move away from the initial position because the space around has already occupied by others. This is called jamming. In a glass dynamics, we usually choose the gross variables as the fluctuations of density of particles and the corresponding current.

We consider a classical N -particle system, where the Hamiltonian is given by [28]

$$H = \sum_i \frac{\mathbf{p}_i^2}{2m} + \frac{1}{2} \sum_{i \neq j} \phi(\mathbf{r}_{ij}). \quad (57)$$

Then, the corresponding Liouville operator is given by

$$iL = \frac{1}{m} \sum_i \left(\mathbf{p}_i \cdot \frac{\partial}{\partial \mathbf{r}_i} \right) - \sum_{i \neq j} \left(\frac{\partial \phi(\mathbf{r}_{ij})}{\partial \mathbf{r}_{ij}} \cdot \frac{\partial}{\partial \mathbf{p}_i} \right). \quad (58)$$

In this case, the Fourier transform of the fluctuations of the particle number density $\rho_{\mathbf{k}}(t)$ is given by

$$\delta\rho_{\mathbf{k}}(t) = \sum_i e^{i\mathbf{k}\cdot\mathbf{r}_i(t)} - (2\pi)^3 \rho \delta(\mathbf{k}), \quad (59)$$

where $\rho = N/V$ and V is the volume of our system. In this system, the number of particle is a conserved quantity and the fluctuations of the density should satisfy the equation of continuity

$$\delta\dot{\rho}_{\mathbf{k}}(t) = i|\mathbf{k}|j_{\mathbf{k}}(t). \quad (60)$$

Here, we define the current

$$j_{\mathbf{k}} = \frac{1}{m} \sum_i (\hat{\mathbf{k}} \cdot \mathbf{p}_i) e^{i\mathbf{k}\cdot\mathbf{r}_i}, \quad (61)$$

where $\hat{\mathbf{k}} = \mathbf{k}/|\mathbf{k}|$. Then, we choose the set of the gross variables as follows:

$$\mathbf{A} = \begin{pmatrix} \delta\rho_{\mathbf{k}} \\ j_{\mathbf{k}} \end{pmatrix}. \quad (62)$$

By substituting into Eq. (8), we obtain the evolution equation of \mathbf{A} . We further multiply \mathbf{A}^\dagger from the right and take the

thermal expectation value. Then the evolution equation of the correlation function is given by

$$\frac{\partial}{\partial t} \mathbf{C}(t) = i\Delta \mathbf{C}(t) - \int_0^t d\tau \Xi(\tau) \mathbf{C}(t - \tau), \quad (63)$$

where

$$\mathbf{C}(t) = \langle \mathbf{A}(t) \mathbf{A}^\dagger \rangle_{\text{eq}} = \begin{pmatrix} \langle \delta\rho_{\mathbf{k}}(t) \delta\rho_{-\mathbf{k}} \rangle_{\text{eq}} & \langle \delta\rho_{\mathbf{k}}(t) j_{-\mathbf{k}} \rangle_{\text{eq}} \\ \langle j_{\mathbf{k}}(t) \delta\rho_{-\mathbf{k}} \rangle_{\text{eq}} & \langle j_{\mathbf{k}}(t) j_{-\mathbf{k}} \rangle_{\text{eq}} \end{pmatrix}. \quad (64)$$

Because of the equation of continuity (60), these four correlation functions are not independent. It should be noted that, in the usual discussion of the glass dynamics, we do not consider the thermal equilibrium environment discussed here and the calculation of the memory function is more involved.

The coefficient of the streaming term is given by

$$i\Delta = \begin{pmatrix} 0 & ik \\ \frac{ik}{m\beta S(\mathbf{k})} & 0 \end{pmatrix}, \quad (65)$$

where the static structure factor is $S(\mathbf{k}) = \frac{1}{N} \langle \rho_{-\mathbf{k}}(0) \rho_{\mathbf{k}}(0) \rangle$. By using Eq. (23), we found that the upper components of the matrix of the memory function vanish,

$$\Xi^L(s) = \begin{pmatrix} 0 & 0 \\ \Xi_{21}^L(s) & \Xi_{22}^L(s) \end{pmatrix}. \quad (66)$$

Because of $[\xi_i(t), A_j^*] = 0$, the noise term disappears.

We will concentrate on the element in the lower column of the matrix. Then, we can obtain the following two equations:

$$\begin{aligned} \frac{\partial^2}{\partial t^2} F(k, t) + \frac{k^2}{m\beta S(\mathbf{k})} F(k, t) + \int_0^t d\tau \left[\Xi_{21}(\tau) (ik) F(k, t - \tau) \right. \\ \left. + \Xi_{22}(\tau) \frac{\partial}{\partial t} F(k, t - \tau) \right] = 0. \end{aligned} \quad (67)$$

$$\begin{aligned} \frac{\partial^3}{\partial t^3} F(k, t) + \frac{k^2}{m\beta S(\mathbf{k})} \frac{\partial}{\partial t} F(k, t) + \int_0^t d\tau \left[\Xi_{21}(\tau) (ik) \frac{\partial}{\partial t} F(k, t - \tau) \right. \\ \left. + \Xi_{22}(\tau) \frac{\partial^2}{\partial t^2} F(k, t - \tau) \right] = 0. \end{aligned} \quad (68)$$

Here, we introduce the following function:

$$F(k, t) = \frac{1}{N} \langle \delta\rho_{\mathbf{k}}(t) \delta\rho_{-\mathbf{k}} \rangle_{\text{eq}}. \quad (69)$$

From the consistency of the two equations, one can find that $\Xi_{21}(\tau)$ should vanish. As a matter of fact, this is shown by using the following exact relation:

$$(\xi_i(t), \xi_j) = (\xi_j(-t), \xi_i)^* = [\Xi(t) \cdot (\mathbf{A}, \mathbf{A}^\dagger)]_{ij}. \quad (70)$$

This relates the memory function and the noise term and is called the fluctuation-dissipation theorem of the second kind [26]. From this relation, one can show that $\Xi_{21}(t)$ also should disappear when $\Xi_{12}(t)$ vanishes. The correlation functions, then, should satisfy the following relation:

$$\begin{aligned} \Xi_{12}^L(s) \propto \{ \dot{\mathbf{X}}_{21}^L(s) - [\dot{\mathbf{X}}(0) \dot{\mathbf{X}}^L(s)]_{21} \} [1 + \dot{\mathbf{X}}_{22}^L(s)] - \{ \dot{\mathbf{X}}_{22}^L(s) \\ - [\dot{\mathbf{X}}(0) \dot{\mathbf{X}}^L(s)]_{22} \} \dot{\mathbf{X}}_{21}^L(s) = 0. \end{aligned} \quad (71)$$

By using this relation, we can simplify the remaining memory term

$$\begin{aligned} \Xi_{22}^L(s) &= - \frac{\dot{\mathbf{X}}_{22}^L(s) - [\dot{\mathbf{X}}(0) \dot{\mathbf{X}}^L(s)]_{22}}{1 + \dot{\mathbf{X}}_{22}^L(s)} \\ &= - \frac{s^2 F^L(k, s) - s S(\mathbf{k}) + k^2 F^L(k, s) / [m\beta S(\mathbf{k})]}{s F^L(k, s) - S(\mathbf{k})}. \end{aligned} \quad (72)$$

This is the result obtained by using our expression of the memory function (23).

In the case discussed here, however, all correlation functions are expressed by the unique correlation function $F(k, t)$ and we do not need to use Eq. (23) to calculate the memory function. From Eq. (67), the Laplace transform of the equation is

$$\begin{aligned} s^2 F^L(k, s) - s F(k, 0) - \dot{F}(k, 0) &= - \frac{k^2}{m\beta S(\mathbf{k})} F^L(k, s) - \Xi_{22}^L(s) \\ &\quad \times [s F^L(k, s) - F(k, 0)]. \end{aligned} \quad (73)$$

The expression of the memory function that is obtained by solving the equation above is same as Eq. (72). This means the consistency of our formula.

VIII. SHEAR VISCOSITY OF CAUSAL DISSIPATIVE HYDRODYNAMICS IN $N=2$ FORM

In the previous derivation of the causal shear viscosity, we assumed that the macroscopic motion can be projected onto the space spanned by the unique gross variable $T^{0x}(k_y)$. If this assumption is correct, the memory function converges to a constant rapidly and we can define the causal shear viscosity coefficient and the relaxation time as was done in Sec. V. The memory functions are calculated so far for several examples and the behaviors are consistent with this assumption.

However, there is another suggestion for the definition of the projection operator. In the wake of the discussion of the extended thermodynamics [6], Ichiyanagi proposed that the Mori projection operator should be defined by using not only usual hydrodynamic variables but also the corresponding currents [44], although any calculable formula was not given. In this section, we rederive the formula for the causal shear viscosity coefficient following his idea.

The set of the gross variables are given by

$$\mathbf{A} = \begin{pmatrix} T^{0x}(k_y) \\ T^{yx}(k_y) \end{pmatrix}. \quad (74)$$

By substituting into Eq. (8), we have

$$\frac{\partial}{\partial t} T^{0x}(k_y, t) = -ik_y T^{yx}(k_y, t), \quad (75)$$

$$\frac{\partial}{\partial t} T^{yx}(k_y, t) = -ik_y R_{k_y} T^{0x}(k_y, t) - \int_0^t d\tau \Xi_{22}(\tau) T^{yx}(k_y, t - \tau), \quad (76)$$

where

$$R_{k_y} = (T^{yx}(k_y), T^{yx}(-k_y))(T^{0x}(k_y), T^{0x}(-k_y))^{-1}. \quad (77)$$

The memory function is given by

$$\Xi_{22}^L(s) = -\frac{\ddot{\mathbf{X}}_{22}^L(s) + ik_y R_{k_y} \dot{\mathbf{X}}_{12}^L(s)}{1 + \dot{\mathbf{X}}_{22}^L(s)} = -\frac{\ddot{\mathbf{X}}_{11}^L(s) + \ddot{\mathbf{X}}_{22}^L(s)}{1 + \dot{\mathbf{X}}_{22}^L(s)} \quad (78)$$

with the Laplace transforms of the following correlations:

$$\mathbf{X}(t) = \begin{pmatrix} (T^{0x}(k_y, t), T^{0x}(-k_y)) & (T^{0x}(k_y, t), T^{yx}(-k_y)) \\ (T^{yx}(k_y, t), T^{0x}(-k_y)) & (T^{yx}(k_y, t), T^{yx}(-k_y)) \end{pmatrix} \\ \times \begin{pmatrix} (T^{0x}(k_y), T^{0x}(-k_y))^{-1} & 0 \\ 0 & (T^{yx}(k_y), T^{yx}(-k_y))^{-1} \end{pmatrix}, \quad (79)$$

$$\dot{\mathbf{X}}(t) = \frac{\partial}{\partial t} \mathbf{X}(t), \quad (79)$$

$$\ddot{\mathbf{X}}(t) = \frac{\partial^2}{\partial t^2} \mathbf{X}(t). \quad (80)$$

Here, we omitted the noise term.

It should be noted that we still define Kubo's canonical correlation by Eq. (12), where the expectation value is calculated by the usual thermal equilibrium state. This is different from the original idea of Ichiyanagi and the extended thermodynamics, where the concept of the thermodynamic variables are extended and hence we have to use a nonequilibrium state to calculate the expectation value. However, when we restrict ourselves to the nonequilibrium states whose deviation from equilibrium is still small, then the expectation values can be evaluated at the equilibrium, since the effect of the nonequilibrium expectation should be higher order and we assume that it is negligible.

Equation (75) is the equation of continuity. If we can derive the causal dissipative hydrodynamics from the Heisenberg equation of motion, Eq. (76) should be reduced to the telegraph equation

$$\frac{\partial}{\partial t} T^{yx}(k_y, t) = -\frac{\eta^{NN}}{2\tau} (ik_y) u^x(k_y, t) - \frac{1}{\tau} T^{yx}(k_y, t). \quad (81)$$

It should be noted that when we combine this equation with the equation of continuity, we can reproduce Eq. (45). To obtain the telegraph equation from Eq. (76), we assume that the memory function Ξ_{22} is Markovian; the memory function quickly vanishes with time,

$$\Xi_{22}(t) = \frac{2}{\tau_{k_y}} \delta(t), \quad (82)$$

where

$$\frac{1}{\tau_{k_y}} = \int_0^\infty d\tau \Xi_{22}. \quad (83)$$

By substituting them into Eq. (76), we obtain

$$\frac{\partial}{\partial t} T^{yx}(k_y, t) = -ik_y w R_{k_y} u^x(k_y, t) - \frac{1}{\tau_{k_y}} T^{yx}(k_y, t). \quad (84)$$

By comparison with the telegraph equation, we identify the causal shear viscosity coefficient and the relaxation time as follows:

$$\tau = \lim_{k_y \rightarrow 0} \tau_{k_y}, \quad (85)$$

$$\eta^{NN} = \lim_{k_y \rightarrow 0} 2R_{k_y} \tau. \quad (86)$$

Note that these formula look different from those in the previous section, formulas (47) and (46). We know that the two approaches ($n=1$ and $n=2$) shown in this paper are completely equivalent if no approximation is introduced, that is, the coupled equations (75) and (76) gives exactly same result as Eq. (28). We even checked the consistency of the two approaches by solving the coupled harmonic oscillator model which is exactly solvable [26].

To use formulas (85) and (86), we have to calculate three correlation functions, while we need two correlation functions in formulas (47) and (46). Thus we should usually use formulas (47) and (46) to estimate the causal shear viscosity coefficient in causal dissipative hydrodynamics. However, when the calculated memory function does not satisfy the condition (38), we have to use formulas (85) and (86).

IX. CONCLUDING REMARKS

In this paper, we derived the general expression of the memory function extending the result of Ref. [24]. By using the expression, we define the shear viscosity coefficient and the corresponding relaxation time of the causal dissipative hydrodynamics. Our formula is the generalization of the GKN formula because, when the Q approximation is justified in the low momentum limit, the GKN formula is reproduced.

Phenomenologically, the causal hydrodynamics is derived by introducing the memory effect to the relation between irreversible currents and thermodynamic forces. Thus the vanishing relaxation time limit $\tau \rightarrow 0$ corresponds to the limit of the Newtonian fluid and hence the causal dissipative hydrodynamics is reduced to the relativistic Navier-Stokes equation (the Landau-Lifshitz theory). Thus it is sometimes expected that the causal shear viscosity coefficient is still approximately given by the calculation of the GKN formula, when the relaxation time is not large. However, this expectation is not trivial. As was discussed in this paper, the new formula reduced to the GKN formula in the Q approximation. In this limit, as is shown in Eq. (55), the causal shear viscosity coefficient η^{NN} vanishes and the expression of the relaxation time τ is reduced to that of the shear viscosity coefficient in the GKN formula. That is, what is approximately given by the GKN formula is not the causal shear viscosity coefficient but the relaxation time.

By using the idea of the AdS/CFT (anti-de Sitter/conformal field theory) correspondence in the string theory, we can calculate the correlation function of the energy-momentum tensor in $\mathcal{N}=4$ supersymmetric Yang-Mills theory [45]. From this result, we obtain $\eta/s=1/(4\pi)$ with s being the entropy density and many people expect that this gives the minimum of the shear viscosity coefficient of the relativistic fluid of quarks and gluons. It should be, however, noted that the η here is not η^{NN} but η^{NS} , that is, this discussion is true only for Newtonian fluids, because, to derive the result, the expression of the GKN formula of the shear viscosity coefficient is used. Thus, when we discuss the causal dissipative hydrodynamics, we cannot use this value as the limit of the causal shear viscosity coefficient. The lower bound of the shear viscosity coefficient may exist even for the causal hydrodynamics. This will be predicted by using our formula instead of the GKN formula.

There are several approaches to derive the relativistic hydrodynamics consistent with causality. However, as far as we know, the telegraph equation plays an essential role to solve the problem of acausality in all theories, and the difference of the theories comes from the nonlinear terms. Thus the formula discussed here is applicable even for other causal dissipative hydrodynamics, the Israel-Stewart theory [3]. See Ref. [10], for more discussions about the relationship between different theories. The effect of non-linearity, in general, can change the coefficients of the linear terms. To discuss the effect of nonlinearity to the transport coefficients, we have to consider the nonlinear response [46]. In the projection operator method, this is implemented by generalizing the projection operator including nonlinear terms. However, the quantitative effect has not been known so far.

On the other hand, the telegraph equation may be not unique solution of the problem of acausality in hydrodynamics. For example, there are different approaches to solve this problem in diffusion processes [47]. However, to our best knowledge, there is no formulation of causal dissipative hydrodynamics in these alternative scenarios. It should also worth mentioning that we have not so far encountered any problem in implementing numerical simulations of the causal dissipative hydrodynamics [8,10].

It should be mentioned that the projection operator approach discussed here and the usual linear response theory do not have explicit Lorentz covariance, because we introduce a thermal equilibrium background. That is, when the transport coefficients of relativistic fluids are calculated, we assume the existence of a local rest frame where the dynamics of macroscopic quantities are determined in a non-relativistic way, together with an appropriate boundary condition.

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APPENDIX A: THE GREEN-KUBO-NAKANO FORMULA

In this appendix, we give the short review of the GKN formula. As for the calculations of the GKN formula for

relativistic fluid, see, for example, Ref. [48], and references therein.

We consider the system whose Hamiltonian is given by H . By applying an external force, the total Hamiltonian is changed from H to $H+H_{\text{ex}}(t)$, with

$$H_{\text{ex}}(t) = -AF(t), \quad (\text{A1})$$

where A is an operator and $F(t)$ is the c -number external force.

We consider the current J induced by the external force. From the linear response theory, we obtain

$$\langle J \rangle = \int_{-\infty}^t ds \Psi(t-s)F(s), \quad (\text{A2})$$

where the response function is given by

$$\Psi(t) = \int_0^\beta d\lambda \langle \dot{A}(-i\lambda)J(t) \rangle_{\text{eq}}. \quad (\text{A3})$$

This is the exact result in the sense of the linear approximation. This formula, also, is called the GKN formula. However, in particular, when we define transport coefficients of hydrodynamics, we do not use this expression.

In these cases, first, we assume the linear relation between currents and the external force, $J(t)=D_{\text{GKN}}F(t)$ with the transport coefficient D_{GKN} . The formula to define the expression D_{GKN} is the GKN formula which is discussed in this paper. For this, we see that we should ignore the memory effect (time-convolution integral) in Eq. (A2),

$$\langle J \rangle \approx \int_0^\infty ds \Psi(s)F(t). \quad (\text{A4})$$

Then the GKN formula is

$$D_{\text{GKN}} = \int_0^\infty ds \Psi(s). \quad (\text{A5})$$

Thus this formula is applicable only when there is a proportional relation between a current and a force, as in Newtonian fluids. This is the reason why we cannot use the GKN formula to calculate the transport coefficients of the causal dissipative hydrodynamics.

In principle, it is possible to derive the transport coefficients of the causal dissipative hydrodynamics from Eq. (A2). Instead of $J(t)=D_{\text{GKN}}F(t)$, we assume the following telegraph equation:

$$\partial_t J(t) = -\frac{1}{\tau_R}J(t) + \frac{D}{\tau_R}F(t). \quad (\text{A6})$$

From Eq. (A2), we can derive the following equation:

$$\partial_t J(t) = \Psi(0)F(t) + \int_0^\infty ds \partial_s \Psi(s)F(t). \quad (\text{A7})$$

In the second term, we ignore the time-convolution integral. We further assume the GKN formula to reexpress the first term. Then we finally obtain

$$\partial_t J(t) = \frac{\Psi(0)}{D_{\text{GKN}}} J(t) + \int_0^\infty ds \partial_s \Psi(s) F(t). \quad (\text{A8})$$

By comparing this equation with Eq. (A6), we can derive the expression of D and τ_R .

Exactly speaking, we considered here the current induced by the external force. However, the shear viscosity is induced not by the external force but by the difference of the boundary conditions. Thus the discussion is not applicable to the problems discussed in this paper.

APPENDIX B: THE DERIVATION OF EQ. (22)

In this appendix, we derive Eq. (22). By using Eqs. (16) and (19), we obtain

$$\Xi_{ij}(t) = -\theta(t) \sum_k (iLe^{iLt} \mathcal{B}(t) QiLA_i, A_k^\dagger) (A, A^\dagger)_{kj}^{-1}. \quad (\text{B1})$$

The first three terms can be calculated as follows:

$$(iLe^{iLt} QiLA_i, A_j^\dagger) (A_j, A_j^\dagger)^{-1} = \ddot{\mathbf{X}}_{ij}(t) - [\dot{\mathbf{X}}(0) \dot{\mathbf{X}}(t)]_{ij}, \quad (\text{B2})$$

$$\left(iLe^{iLt} (-i) \int_0^t ds e^{-iLs} PLe^{iLs} QiLA_i, A_j^\dagger \right) (A_j, A_j^\dagger)^{-1} = - \int_0^t ds \{ [\ddot{\mathbf{X}}(s) \dot{\mathbf{X}}(t-s)]_{ij} - [\dot{\mathbf{X}}(0) \dot{\mathbf{X}}(s) \dot{\mathbf{X}}(t-s)]_{ij} \}, \quad (\text{B3})$$

$$\begin{aligned} & \left(iLe^{iLt} \int_0^t ds_1 e^{-iLs_1} PLe^{iLs_1} \int_0^{s_1} ds_2 e^{-iLs_2} PLe^{iLs_2} QiLA_i, A_j^\dagger \right) (A_j, A_j^\dagger)^{-1} \\ &= \int_0^t ds_1 \int_0^{s_1} ds_2 \{ [\ddot{\mathbf{X}}(s_2) [\dot{\mathbf{X}}(s_1 - s_2) \dot{\mathbf{X}}(t - s_1)]_{ij} - [\dot{\mathbf{X}}(0) \dot{\mathbf{X}}(s_2) \dot{\mathbf{X}}(s_1 - s_2) \dot{\mathbf{X}}(t - s_1)]_{ij} \}. \end{aligned} \quad (\text{B4})$$

In short, the n th order term is given by

$$\begin{aligned} & \left(iLe^{iLt} (-i)^n \int_0^t ds_1 \cdots \int_0^{s_{n-1}} ds_n \check{L}^P(s_1) \cdots \check{L}^P(s_n) QiLA_i, A_j^\dagger \right) (A_j, A_j^\dagger)^{-1} \\ &= (-1)^n \int_0^t ds_1 \cdots ds_n [\ddot{\mathbf{X}}(s_n) \dot{\mathbf{X}}(s_{n-1} - s_n) \cdots \dot{\mathbf{X}}(s_1 - s_2) \dot{\mathbf{X}}(t - s_1)]_{ij} \\ & \quad - (-1)^n \int_0^t ds_1 \cdots \int_0^{s_{n-1}} ds_n [\mathbf{X}(0) \dot{\mathbf{X}}(s_n) \dot{\mathbf{X}}(s_{n-1} - s_n) \cdots \dot{\mathbf{X}}(s_1 - s_2) \dot{\mathbf{X}}(t - s_1)]_{ij}. \end{aligned} \quad (\text{B5})$$

By using this result, we can calculate

$$\begin{aligned} \sum_j (iLQe^{iLQs} iLA_1, A_j^\dagger) (A_j, A_j^\dagger)^{-1} &= [\ddot{\mathbf{X}}(t)]_{11} - [\dot{\mathbf{X}}(0) \dot{\mathbf{X}}(t)]_{11} - \int_0^t d\{ [\ddot{\mathbf{X}}(s) \dot{\mathbf{X}}(t-s)]_{11} - [\dot{\mathbf{X}}(0) \dot{\mathbf{X}}(s) \dot{\mathbf{X}}(t-s)]_{11} \} \\ & \quad - \int_0^t d_1 \int_0^{s_1} ds_2 \{ [\ddot{\mathbf{X}}(s_2) \dot{\mathbf{X}}(s_1 - s_2) \dot{\mathbf{X}}(t - s_1)]_{11} - [\dot{\mathbf{X}}(0) \dot{\mathbf{X}}(s_2) \dot{\mathbf{X}}(s_1 - s_2) \dot{\mathbf{X}}(t - s_1)]_{11} \} + \cdots. \end{aligned} \quad (\text{B6})$$

In short, the Laplace transform of Eq. (22) is given by

$$[\Xi^L(s)]_{11} = - \left(\ddot{\mathbf{X}}^L(s) \frac{1}{1 + \dot{\mathbf{X}}^L(s)} - \dot{\mathbf{X}}(0) \dot{\mathbf{X}}^L(s) \frac{1}{1 + \dot{\mathbf{X}}^L(s)} \right)_{11}. \quad (\text{B7})$$

The other components of the matrix are calculated in the same way.

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- [42] Exactly speaking, there can exist the difference of the factor between the result of the POM and the GKN formula, because the linear response theory sometimes has an ambiguity for the definition of the transport coefficients. This is clear in the case of the definition of the diffusion coefficient. It is known that the POM can predict the correct expression, although the additional discussion is necessary to fix the proportional factor in the linear response theory [23].
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